On Goal-Directed Provability in Classical Logic

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1 Introduction

One of the fundamental features of logic programming is the notion of proof search, in that in a logic programming setting, computation is interpreted as the search for a proof of a given formula from a given set of assumptions. Hence from the viewpoint of logic, logic programming is concerned not just with provability, but a stronger notion, often referred to as *goal-directed* provability, in that the structure of the goal should determine the structure of the proof.

The best known characterization of goal-directed provability is the notion of a *uniform proof* in the sequent calculus for intuitionistic logic [9]. An intuitionistic proof is uniform if for any sequent in the proof for which the succedent is not an atom, the rule used to derive the sequent is the introduction rule for the principle connective of the succedent. In this way, the sequence of steps in the search for a uniform proof is determined by the structure of the succedent, i.e. the goal. This particular notion of goal directed provability has been studied in various ways, [9, 8, 3], and has also lead to similar investigations in linear logic [4].

Whilst the connections between intuitionistic logic and computation are various and well known, from the point of view of proof search, there seems to be no inherent reason why we cannot use classical logic to find a logic programming language. In particular, it should be noted that the completeness of uniform proofs in intuitionistic logic requires that there be no disjunctions in the heads of rules. However, the so-called *disjunctive logic programs* [10] allow such a possibility, suggesting that the use of classical, rather than intuitionistic, logic is necessary in this case. Indeed, the coincidence of classical, intuitionistic and minimal logics on the Horn clause fragment means that it is not always clear what is the optimal choice of logic for larger fragments.

In this paper we explore the possibilities for a notion of goal-directed proof in classical logic. The technical point to consider is how to deal with the multiple-conclusioned nature of classical sequents, i.e. that classical succedents may contain more than one formula. This means that there may be more than one "candidate" right rule, as there may be several non-atomic formulae in the succedent, and so a choice has to be made as to which formula is to be reduced in the next step. The question of whether this choice may be free or restricted is one of the key decisions to be made. The free choice, i.e. that the order in which the formulae are reduced does not matter, will clearly constrain the logic programming language more than the restricted, one, and is arguably more declarative; on the other hand, the weaker notion is arguably more goal-directed, and there is no obvious reason to insist on the stronger version. We will refer to the free choice as *right-reductive* proofs, and to the restricted one as *right-directed* proofs. Thus there seems to be more than one notion of goal-directed proof in classical logic, and clearly the corresponding logic programming languages may differ according to which class of proofs is used. However, as we shall see, there are do not seem to be any "interesting" languages for which the weaker notion is complete but the stronger one is not, and so it appears the stronger version (which requires that all right rules permute over each other) is the more useful notion.

We discuss these various notions of goal-directed provability, and show how they may be used to determine classes of formulae for which there are complete.

It should be noted that here we are dealing with the provability of *sequents*, and not of formulae. Hence the well known normalisation properties of classical logic,

such as that every formula is classically equivalent to one in disjunctive normal form, are of little relevance here. For example, consider the sequent $\exists x p(x) \longrightarrow \exists x p(x)$. For this sequent to have a goal-directed proof (according to either conception) we would require that the sequent $\exists x p(x) \longrightarrow p(t)$ be provable for some term t, but clearly there is no such term. Thus we conclude that there is no goal-directed proof of this sequent, regardless of whether or not a classically equivalent sequent such as $\forall x \neg p(x) \longrightarrow \forall x \neg p(x)$ has a goal-directed proof.

It should be noted that the results presented here are not particularly difficult to establish, and many are well known. It is the application of these results to the problem of determining what classical logic has to offer logic programming which is the main focus of this paper, and in particular to determine the completeness properties of a certain kind of proof search. This makes it seem most appropriate to use the sequent calculus as our main investigative tool, rather than other proof systems such as natural deduction, Hilbert-type systems or tableau systems, or model-theoretic means.

2 Notions of Classical Goal-Directedness

A proof-theoretic characterization of *goal-directed* provability is the notion of a *uniform proof*, which has been studied extensively in the intuitionistic case [8, 9, 3]. Intuitively, a uniform proof requires that the right rules be used closer to the root of the proof tree than the left rules.

Definition 2.1 (Uniform intuitionistic proofs) *An intuitionistic proof Ξ is* uniform *if for every sequent $\Gamma \longrightarrow F$ in Ξ in which F is non-atomic formula, the rule used to infer $\Gamma \longrightarrow F$ is the right rule for the principal connective of F.*

Thus uniform proofs have a "local" shape of left rules above right rules. Note that not all provable sequents have such proofs. For example, the provable sequent $p \vee q \longrightarrow p \vee q$ does not have a uniform (intuitionistic) proof. Hence it seems natural to investigate the completeness properties of this class of proofs. In [9] it is shown that uniform proofs are complete for a class of formulae known as *hereditary Harrop formulae*, in that if Γ and F obey a certain syntactic restriction, then $\Gamma \vdash_I F$ iff $\Gamma \longrightarrow F$ has a uniform proof, where $\vdash_I$ denotes intuitionistic provability. The restriction is that Γ be a set of *definite* formulae and F be a *goal* formula. These classes of formulae are given below.

Definition 2.2 *We define the class of* definite *formulae and* goal *formulae by the grammar:*

$$D ::= A \mid \forall x D \mid D \wedge D \mid G \supset A$$
$$G ::= A \mid \exists x G \mid \forall x G \mid G \wedge G \mid G \vee G \mid D \supset G$$

where A is an atom.

Definite formulae are essentially those in which there is no positive occurrence of $\exists x F$ or $F_1 \vee F_2$ (and no negations anywhere). It is known that the class formulae in which formulae of the form $G \supset D$ are allowed is no more powerful, in that given a set of formulae in the larger class there is an intuitionistically equivalent set of formulae in the smaller one. There is some evidence to support the contention that this is the largest class of formulae for which (intuitionistic) uniform provability is complete [3].

As mentioned above, in order to study goal-directed provability in classical logic, it is necessary to consider whether proofs should be right-reductive or only right-directedness. In addition, there is the question of whether these properties apply

to succedents which contain atomic formulae or not (i.e. whether it is necessary to reduce a succedent until it contains only atoms before considering left rules).

We consider first the question of right-reductiveness and right-directedness. From the point of view of proof search, right-directedness allows the possibility that there may be some choices of right rules which will not work, and so there is some restriction on the choice of rule to use, whereas right-reductiveness requires that any rule may be used. Clearly both approaches reduce to the same thing when there is only one formula in the succedent, and hence either may be considered a reasonable way to use the notion of uniform provability in intuitionistic logic as a guide for a notion of goal-directed provability in classical logic.

Definition 2.3 (Right-directed classical proofs) *A classical proof Ξ is* right-directed *if for every non-initial sequent $\Gamma \longrightarrow \Delta$ in the proof for which Δ contains a non-atomic formula, $\Gamma \longrightarrow \Delta$ is the conclusion of a right rule.*

Definition 2.4 (Right-reductive classical proofs) *A classical proof Ξ is* right-reductive *if it is right-directed and for every sequent $\Gamma \longrightarrow \Delta$ in the proof for which Δ contains a non-atomic formula, there is a right-directed proof in which the principle connective of the non-atomic formula is introduced in the previous step.*

As noted above, the sequent $\exists x p(x) \longrightarrow \exists x p(x)$ does not have a right-directed proof, and hence does not have a right-reductive proof either. On the other hand, the sequent $p, q, r(t) \longrightarrow p \vee q, \exists x r(x)$ clearly has a right-reductive proof and hence a right-directed proof. However, there are sequents which have a right-directed proof but not a right-reductive proof — for example, consider the following sequent (which was suggested by an anonymous referee): $\forall x\, p(x) \vee q(x) \longrightarrow \forall x\, q(x), \exists x\, p(x)$. This sequent is provable, by first applying the $\forall$-R rule "backwards", so that the sequent is reduced to $\forall x\, p(x) \vee q(x) \longrightarrow q(y), \exists x\, p(x)$, and then by the $\exists$-R rule to $\forall x\, p(x) \vee q(x) \longrightarrow q(y), p(y)$, which may be derived by an instance of $\forall$-L and an instance of $\vee$-L. Hence it has a right-directed proof. However, there is no term t such that $\forall x\, p(x) \vee q(x) \longrightarrow \forall x\, q(x), p(t)$, and hence it is not possible to apply the $\exists$-R rule before the $\forall$-R rule. The reason is the side condition on the $\forall$-R rule necessitates that the $\forall$-R rule is used closer to the root of the proof than the $\exists$-R rule.

3 Right-reductive Proofs

Hence the aim now is to determine classes of formulae for which each notion of goal-directed provability is complete. We first consider right-reductive proofs. Clearly any such class of formulae will need to exclude the sequent $\exists x p(x) \longrightarrow \exists x p(x)$, and as we saw above, we need to exclude the sequent $\forall x p(x) \vee q(x) \rightarrow \forall x q(x), \exists x p(x)$ as well. In particular, we cannot hope to maintain both quantifiers in the succedent, as this may require the "delaying" of the use of the rule for the existential quantifier until after the application of the corresponding rule for the universal quantifier. Hence, we eliminate the existential quantifier from antecedents and the universal quantifier from succedents. Thus we arrive at the class of formulae below.

Definition 3.1 *We define the class of* existential-free *formulae and* universal-free *formulae by the grammar:*

$$D ::= A \mid \forall x D \mid D \wedge D \mid D \vee D \mid \neg G \mid G \supset D$$
$$G ::= A \mid \exists x G \mid G \wedge G \mid G \vee G \mid D \supset G \mid \neg D$$

where A is an atom.

Note that this class of formulae does not quite generalise that of hereditary Harrop formulae, as $\forall x p(x)$ is a hereditary Harrop formula goal, but is not a universal-free formula. Another difference between the two classes is that universal-free formulae may contain positive occurrences of disjunctions. This results in "disjunctive" answers to existentially quantified goals, as given in the result below.

Lemma 1. *Let Γ be a set of existential-free formulae, and Δ be a set of universal-free formulae. Then $\Gamma \vdash_{LK} \exists x F, \Delta \Leftrightarrow \Gamma \vdash_{LK} F[t_1/x], \ldots, F[t_n/x], \Delta$ for some terms $t_1, \ldots, t_n$.*

Note that the "disjunctive" nature of this result is necessary, in that it is not necessarily true that if $\Gamma \longrightarrow \exists x F, \Delta$ is provable, then so is $\Gamma \longrightarrow F[t/x], \Delta$ for some term t. For example, consider the provable sequent $p(a) \vee p(b) \longrightarrow \exists x\, p(x)$. Whilst we have that $p(a) \vee p(b) \vdash_{LK} p(a), p(b)$, we do not have that $p(a) \vee p(b) \vdash_{LK} p(t)$ for any term t. Hence given the program $p(a) \vee p(b)$, the best we can do for the query $\exists x\, p(x)$ is to return the disjunctive answer that x is either a or b. Thus disjunctive answers may be thought of as succedents with multiple conclusions. Note that this may be thought of as a way of dealing with contraction on the right for existentially quantified formulae.

This result suggests that for this class of formulae, a slightly different form of the $\exists$-R rule may be used, in which the formula $F[t/x]$ in the premiss is replaced by $F[t_1/x], \ldots F[t_n/x]$ where $t_1, \ldots t_n$ are terms. We refer to this version of LK with the $\exists$-R rule replaced by this rule as LK'. It should be clear that $\exists$-R is a special case of this rule, and so any LK-proof is an LK'-proof. Moreover, it should also be clear that the $\exists$'-R rule is a derived rule of LK, and hence provability in LK and LK' coincide.

It is then trivial to show the following proposition.

Proposition 2 *Let $\Gamma \longrightarrow \Delta$ be a universal-free sequent. Then $\Gamma \vdash_{LK'} \Delta \Leftrightarrow \Gamma \longrightarrow \Delta$ has a right-reductive proof in LK'.*

It should be noted that universal-free formulae contain negations and disjunctions, but do not allow universal quantifiers to occur positively in the succedent. A question that may be asked is whether it is possible to obtain a similar result in which such universal quantifiers are allowed, but some other restriction is made. It is not hard to see that such a restriction would need to omit negations and implications in the succedent: consider the provable sequents $\vdash \forall x \neg p(x), \exists x p(x)$ and $\vdash \forall x(p(x) \supset q), \exists x p(x)$. In addition, we must exclude the $\forall x\, p(x) \vee q(x) \longrightarrow \forall x\, q(x), \exists x p(x)$ for right-reductive proofs. As the main motivation is to maintain both quantifiers in the succedent, this would indicate that disjunctions need to be disallowed in antecedents. This leads us to the class of formulae below.

Definition 3.2 *We define the class of* flat definite *formulae and* flat goal *formulae by the grammar:*

$$D ::= A \mid \forall x D \mid D \wedge D \mid \neg G \mid G \supset D$$
$$G ::= A \mid \exists x G \mid \forall x G \mid G \wedge G \mid G \vee G$$

where A is an atom. A sequent $\Gamma \longrightarrow \Delta$ is a flat definite *sequent if Γ is a set of flat definite formulae and Δ is a set of flat goal formulae.*

An interesting property of flat definite formulae is that they are somewhat constructive, as indicated in the result below.

Lemma 3. *Let $\Gamma \longrightarrow \Delta$ be a flat definite sequent. Then $\Gamma \vdash_{LK} \Delta \Leftrightarrow \Gamma \vdash_{LK} F$ for some $F \in \Delta$.*

Note that this result requires the absence of $\vee$ in the antecedent. [1] Note also that this result does not hold for existential-free formulae, as the sequent $\vdash p, p \supset q$ is provable, but neither of the formulae in the succedent is itself provable. Hence deduction with flat definite formulae is essentially intuitionistic, and in particular it is not hard to see that we may use the intuitionistic form of the $\vee$-R rule.

It is then straightforward to show the completeness of right-reductive proofs for this class of formulae.

Proposition 4 *Let $\Gamma \longrightarrow \Delta$ be a flat definite sequent. Then $\Gamma \vdash_{LK} \Delta \Leftrightarrow \Gamma \longrightarrow \Delta$ has a right-reductive proof.*

As noted above, hereditary Harrop formulae do not really work as a logic programming language in the classical setting. This is due to the different properties of classical implication. For example, the following is a definite sequent which is provable classically but not intuitionistically.

$$\vdash p \vee (p \supset q)$$

Hence it would seem that the best analogue of hereditary Harrop formulae in classical logic is flat definite formulae.

4 Right-Directed Proofs

In the previous sections we explored the completeness properties of right-reductive proofs. Now we turn to right-directed proofs. Clearly the only classes of formulae for which this notion of proof will be interesting are those for which right-reductive proofs are not complete, but right-directed proofs are. Hence we will only concern ourselves with the classes of proofs in which both quantifiers are allowed in succedents. However, whilst there are sequents which have a right-directed proof but not a right-reductive one, allowing both quantifiers means that there are some sequents for which the goal-directed strategy will not work. For example, consider the sequent $\Gamma, (\forall x q(x)) \supset r \rightarrow r, \exists x p(x)$. Clearly we may reduce this by using the $\supset$-L rule, to get an initial sequent and the sequent $\Gamma, (\forall x q(x)) \supset r \rightarrow \forall x q(x), r, \exists x p(x)$. Hence, we cannot reduce the existential quantifier in the original sequent, as we have seen that there are occasions on which the processing of the existential quantifier must be delayed until after the universal quantifier has been reduced. Thus we cannot have the completeness of goal-directed proofs, as we may have to "delay" the processing of existential quantifiers in this manner. Similar remarks apply to the sequent $\Gamma, \neg \forall x p(x) \rightarrow \exists x p(x)$.

Hence not only will we need to exclude existential quantifiers in antecedents, we will need to exclude implications and negations in antecedents as well. This leads us to the class of formulae below.

Definition 4.1 *We define the class of* fact *formulae and* fact goal *formulae by the grammar:*

$$D ::= A \mid \forall x D \mid D \wedge D \mid D \vee D$$
$$G ::= A \mid \exists x G \mid \forall x G \mid G \wedge G \mid G \vee G \mid D \supset G \mid \neg D$$

where A is an atom.

We refer to a sequent $\Gamma \longrightarrow \Delta$ where Γ is a set of fact formulae and Δ is a set of fact goal formulae as a fact sequent.

[1] It is possible to extend this result to the case when $\exists$ is allowed in antecedents. However, we have not included this property here, as it is not strictly relevant to the notion of logic programming under discussion.

Using the known permutation properties of the classical rules (as studied by Curry [2] and Kleene [5], it is not hard to show the following proposition.

Proposition 5 *Let $\Gamma \longrightarrow \Delta$ be a fact sequent. Then $\Gamma \vdash_{LK'} \Delta$ iff $\Gamma \longrightarrow \Delta$ has a right-directed proof in LK'.*

As above, we need to use the $\exists'$-R rule to deal with the possibility of disjunctive answer substitutions.

Whilst it is straightforward to prove this result, it is not a particularly interesting one, as the restriction to fact formulae seems to be too strong to be useful. In particular, this class of formula does not include Horn clauses, and hence it is difficult to see how this would be a useful logic programming language. Hence for the notion of right-directed provability to be interesting, a different syntactic restriction will be needed.

We have already seen how, in the presence of implications in the antecedent, we cannot include implications and negations in goal formulae as well as both quantifiers and maintain the completeness of right-directed provability. Hence if we wish to include implications (and negations) in antecedents, the only possibility seems to be excluding disjunctions in antecedents. This gives us the class of formulae below,

$$D ::= A \mid \forall x D \mid D \wedge D \mid \neg G \mid G \supset D$$
$$G ::= A \mid \exists x G \mid \forall x G \mid G \wedge G \mid G \vee G$$

for which, as we have seen, right-reductive proofs are complete. Hence, there seem to be no "interesting" logic programming languages for which right-directed provability is complete, but right-reductive provability is not.

5 Conclusions and Further Work

We have seen how in classical logic there seem to be two notions of goal-directed proof, one of which is stronger than the other. However, it also appears to be the case that all interesting logic programming languages in classical logic are such that right-reductive provability is complete. Hence it would seem that the appropriate notion of goal-directed provability is right-reductive proofs.

As we have seen, it is possible to consider disjunctive logic programs within the framework of the completeness of goal-directed provability by using classical logic rather than intuitionistic logic. The main reason for this is that in the classical case there is a reduction property for disjunctions appearing the succedent regardless of whether or not they appear in the antecedent, but in the intuitionistic case the reduction property relies on the absence of disjunctions in the antecedent. In other words, disjunctive logic programming requires the use of classical logic and not intuitionistic logic in contrast to hereditary Harrop formulae, which require the use of intuitionistic logic rather than classical logic.

Another point for consideration the occurrence of atomic formulae in succedents. In particular, if there is an atom and a compound (i.e. non-atomic) formula in a succedent, should the non-atomic formula "take precedence" or not? It seems that one can argue either way, in that the notion of goal-directedness presumably implies that the presence of compound formulae should be exploited whenever possible, whereas one could also view an atom as a special kind of goal formula, and that the notion of goal-directed provability only applies when no such formulae are present. This is essentially due to the necessity of applying the contraction rule before the $\exists$-R rule in some cases in order to maintain completeness, as all the other connectives do not require this (and in fact all the other right rules are invertible). Hence we may always reduce the succedent to consist only of atoms and formulae with an

existential quantifier outermost. It is interesting to note that a unification method for finding witnesses, such as that of Cerrito [1] or that of Lincoln and Shankar [6] may be useful for further reducing the existentially quantified formulae before applying left rules in such cases.

It should be noted that the notion of goal-directed proof does not imply any restriction when the succedent consists entirely of atoms. In [8] the notion of a *simple* proof is introduced, which, essentially, makes $\supset$-L work as a unary rule. This requires that the formulae in the antecedent be of the form $G \supset A$ rather than $G \supset D$. Hence, in order to study the corresponding notion in classical logic, we will need to transform formulae of the form $G \supset D$ into some equivalent set of simpler formulae. The precise form of such formulae is under investigation. Clearly one may use classical equivalence of $G \supset (D_1 \vee D_2)$ and $(G \supset D_1) \vee (G \supset D_2)$ to good effect in such a simplification.

This also raises the question of multiple-conclusioned sequent calculi for intuitionistic logic. The notion of uniform proof was given only for Gentzen's LJ, in which there can be at most one formula in a succedent. However, it is interesting to speculate how such a notion may be formulated in a multiple-conclusioned system, and whether or not the same uniform proof properties hold in such a system.

It should also be noted that perhaps the notion of goal-directed provability is insufficient per se to determine logic programming languages. As we have seen, it is possible to show that right-directed proofs are complete for fact formulae, but these do not seem to correspond to intuitive notions of what a logic programming language should be. In addition, there are classes of formulae for which right-reductive provability is complete, but appear to be somewhat strange logic programming languages. For example, it is straightforward to show that right-reductive proofs are complete for "existential-free" formulae, given below.

$$D ::= A \mid \exists x D \mid D \wedge D \mid D \vee D \mid \neg G \mid G \supset D$$
$$G ::= A \mid \forall x G \mid G \wedge G \mid G \vee G \mid D \supset G \mid \neg D$$

However, it is difficult to consider this class of formulae in the same light as universal-free formulae.

Thus it seems that the completeness of goal-directed provability is a signpost rather than a definitive criterion in classical logic. For example, when negations are allowed in programs, as in both existential-free formulae and flat definite formulae, the goal may not be actually "relevant" to the proof, such as in the sequent $p, \neg p \longrightarrow q$. Whilst it is true that this sequent has a right-reductive proof, it is clear that there is nothing about this proof that is peculiar to q, and so it seems philosophically difficult to describe this proof as goal-directed. Hence it would seem that stronger restrictions than goal-directed provability need to be placed on the class of proofs in order to determine logic programming languages. It seems probable that McCarty's notion of "perspicuous non-determinism" [7] would be useful here, so that some kind of paradigm of "formulae as instructions" is followed, with a some specific nature of the instruction corresponding to each connective being given by the programmer prior to considerations of the completeness of goal-directed proofs and the like.

6 Acknowledgments

The author is indebted to Dale Miller, for many interesting discussions and weighty observations. Thanks are also due to David Pym, for stimulating discussions of this material, and to some anonymous referees for some helpful comments.

The support of the Australian Research Council and the Centre for Intelligent Decision Systems is gratefully acknowledged.

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