Matrices

(Notes based on Real-Time Rendering by Akenine-Moller, Haines and Hoffman)
Matrices are used in computer graphics for transformations.
Typically, 2×2, 3×3 or 4×4 square matrices are used. In the case of 4×4 the matrices are generally used for *homogenous coordinates*.
An example matrix is

$$\mathbf{M} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$$

A general 2×2, matrix $\mathbf{M}$ is a rectangular (or square) array of numbers

$$\mathbf{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Matrices

Similarly a 3×3, matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

More generally a $p \times q$ matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,q} \\ m_{2,1} & m_{2,2} & \dots & m_{2,q} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ m_{p,1} & m_{p,2} & \dots & m_{p,q} \end{pmatrix} = [m_{i,j}]$$

where each matrix entry i, j is $m_{i,j}$
The notation $[m_{i,j}]$ is a shorter notation for a matrix.

Matrix Addition

Matrices can be added

$$\mathbf{T} = \mathbf{M} + \mathbf{N} = [m_{i,j}] + [n_{i,j}] = [m_{i,j} + n_{i,j}]$$

where $\mathbf{T}$ is the result.

Matrix Multiplication

Matrices can be multiplied

$$
\begin{aligned}
\mathbf{T} &= \mathbf{M} \times \mathbf{N} \\
&= \begin{pmatrix} m_{1,1} & \dots & m_{1,q} \\ . & \dots & . \\ . & \dots & . \\ m_{p,1} & \dots & m_{p,q} \end{pmatrix} \times \begin{pmatrix} n_{1,1} & \dots & n_{1,r} \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ n_{q,1} & \dots & n_{q,r} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^{q} m_{1,i} n_{i,1} & \dots & \sum_{i=1}^{q} m_{1,i} n_{i,r} \\ . & \dots & . \\ . & \dots & . \\ \sum_{i=1}^{q} m_{p,i} n_{i,1} & \dots & \sum_{i=1}^{q} m_{p,i} n_{i,r} \end{pmatrix}
\end{aligned}
$$

Matrix Transpose

The transpose of a matrix has rows and columns interchanged

$$\mathbf{M^T} = [m_{ij}]^T = [m_{ji}]$$

Matrix Inverse

The inverse of a matrix $\mathbf{M}$ is the matrix $\mathbf{M}^{-1}$ such that

$$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$$

The inverse of a matrix may be found in different ways. For a 2×2 matrix $\mathbf{M}$ the inverse $\mathbf{M}^{-1}$ is

$$\mathbf{M}^{-1} = \frac{1}{|\mathbf{M}|}\begin{pmatrix} m_{2,2} & -m_{1,2} \\ -m_{2,1} & m_{1,1} \end{pmatrix}$$

More generally, for higher dimensions, the inverse is defined in terms of the *adjoint* matrix

$$\mathbf{M}^{-1} = \frac{1}{|\mathbf{M}|} adj(\mathbf{M})$$

where $adj(\mathbf{M})$ is the adjoint matrix.

Orthogonal and Orthonormal Matrices

An orthogonal matrix can be thought of as one where each row (or column) taken as a vector is orthogonal to the other rows (or columns).

An orthonormal matrix is an orthogonal matrix where each row (or column) taken as a vector has unit magnitude.

The reason orthonormal matrices are important is that the inverse is the tranpose

$$\mathbf{M}^{-1} = \mathbf{M}^{\mathbf{T}}$$

Orthonormal matrices are used in computer graphics to change from one basis into another, i.e. to change coordinates from coordinate system to another.

Orthonormal matrices are used in bump mapping to transform from *object space* to *tangent space*.