Introduction to Collision Dynamics

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References

[FoP] Halliday, D., Resnick, R. and Walker, J., Fundamentals of Physics

Collision Dynamics

"A *collision* is an isolated event in which two or more bodies (the colliding bodies) exert relatively strong forces on each other for a relatively short time." (from FoP)

Collision dynamics is the dynamics (motion and forces over time) between colliding objects.

Hard Spheres

Collision dynamics are usually quite complicated. They may be simplified by using the following assumptions:

- The objects are spherical particles.
- The particles can be modelled as *points*.
- The collisions are *completely elastic*.
- No forces apply between particles except for the instant they are colliding. In particular, no friction.
- The collisions are instantenous — the spheres are *hard*.

These assumptions are sometimes collectively referred to as hard sphere or billiard ball assumptions.

Center of Mass

The *center of mass* (x_{cm}, y_{cm}, z_{cm}) of a system of particles is a useful quantity in collision dynamics calculations. The center of mass of a system of particles is a point defined as

$$x_{cm} = \frac{1}{M}\sum m_i x_i$$
$$y_{cm} = \frac{1}{M}\sum m_i y_i$$
$$z_{cm} = \frac{1}{M}\sum m_i z_i$$

where M is the sum of the masses of the particles

$$M = \sum m_i$$

Newton's Laws

First
: A body on which no net force acts remains either at rest or moving with constant velocity.

Second
: $\mathbf{F} = m\mathbf{a}$ where $\mathbf{F}$ is the net force, m is the mass and $\mathbf{a}$ is the acceleration.

Third
: For every action there is an equal and opposite reaction.

System of Particles

Newton's laws of motion apply for a system of particles, as much as for a single particle. If $\mathbf{F}_{\mathbf{ext}}$ is the sum of *external* forces operating on the particles, and M the sum of the masses then the second law becomes:

$$\sum \mathbf{F}_{\mathbf{ext}} = M\mathbf{a}_{\mathbf{cm}}$$

External forces are forces arising other than from the effect of one or more of the particles on another.

When $\mathbf{F}_{\mathbf{ext}} = 0$ we get Newton's first law for a system of particles: no force, no change in velocity of the center of mass.

A system can be as small as two particles.

Example

(from FoP, 5th edition)

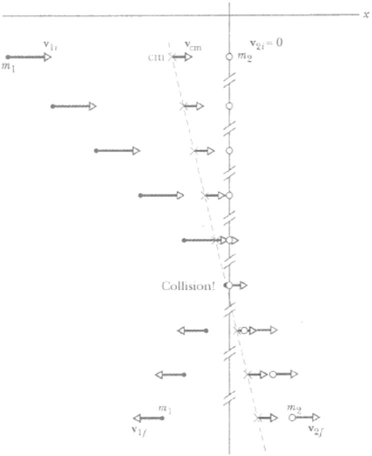


FIGURE 10-8 Some freeze-frames of two bodies undergoing an elastic collision. Body 2 is initially at rest, and $m_2 = 3m_1$. The velocity of the center of mass is also shown. Note that it is unaffected by the collision.

Conservation Principles

The study of mechanics — including dynamics — is often based around finding properties or quantities which are conserved. Two quantities which are conserved under the hard sphere assumption are:

1. Momentum
2. Energy

Momentum

The *momentum* **p**, a vector quantity, of a single particle is defined as

$$\mathbf{p} = m\mathbf{v}$$

The momentum **P** of a collection of particles is

$$\mathbf{P} = \sum m_i \mathbf{v_i}$$

Using momentum, Newton's second law can be written as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$$

And for a system of particles

$$\sum \mathbf{F_{ext}} = \frac{d\mathbf{P}}{dt} = M\frac{d\mathbf{v_{cm}}}{dt} = M\mathbf{a_{cm}}$$

Conservation of Momentum

When there is no force $\mathbf{f} = 0$ Newton's first law gives us the principle of conservation of momentum

$$\mathbf{p} = \textit{constant}$$

Similarly for a system of particles when $\sum \mathbf{F_{ext}} = 0$

$$\mathbf{P} = \textit{constant}$$

Further, because mass is conserved (no particles leaving or joining the system) we get

$$\mathbf{v_{cm}} = \textit{constant}$$

This is true even when the particles of the system are colliding and bouncing off each other — the velocity of the center of mass, and hence the momentum of the center of mass is constant.

Kinetic Energy

Another dynamical quantity which is conserved in elastic collisions is *kinetic energy*.

The kinetic energy of a single particle is

$$K = \frac{1}{2} m v^2$$

Note that kinetic energy is a *scalar* quantity (not a vector quantity).

The total kinetic energy of a system of particles is the sum of the individual particles' kinetic energy

$$K = \frac{1}{2} \sum m_i v_i^2$$

Conservation of Energy

It can be shown that when no external forces act on the sytem of particles that the total kinetic energy of the system does not change — even when collisions occur.
Thus whilst (elastic) collisions might cause the kinetic energy of the individual colliding particles to change, the sum of their kinetic energy does not.

1D Particle Elastic Collision Calculations

Assume two particles with masses m_1 and m_2 are moving with velocities before colliding of v_{1i} and v_{2i}.
What are the velocities v_{1f} and v_{2f} after collision?
We have two equations and two unknowns — solving them similtaneously will give the solution.
Conservation of momentum:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{1}$$

Conservation of (kinetic) energy:

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \tag{2}$$

1D Particle Elastic Collision Calculations (cont)

Solve for v_{1f} and v_{2f}. Rewrite 1 and 2 as

$$m_1(v_{1i} - v_{1f}) = -m_2(v_{2i} - v_{2f}) \quad (3)$$

$$m_1(v_{1i}^2 - v_{1f}^2) = -m_2(v_{2i}^2 - v_{2f}^2) \quad (4)$$

Express 4 as product

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = -m_2(v_{2i} - v_{2f})(v_{2i} + v_{2f}) \quad (5)$$

Dividing 5 by 3, substituting and rearranging gives

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \quad (6)$$

and

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \quad (7)$$

Equal Masses

What happens when the two masses are equal?

1D Particle Inelastic Collision Calculations

In *inelastic collisions* kinetic energy is not conserved. In a (completely) inelastic collision both particles stick together after colliding.

Again assume two particles with masses m_1 and m_2 are moving with velocities before colliding of v_{1i} and v_{2i}. We want to determine velocities v_{1f} and v_{2f} after collision.

The two objects stick together, i.e. move with the same velocity, so there is only one final velocity:

$$v_f = v_{1f} = v_{2f}$$

Conservation of momentum:

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \quad (8)$$

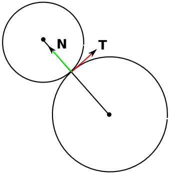
$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} \quad (9)$$

2D Particle Elastic Collision Calculations

In 2D we have more equations and more unknowns.
Momentum — a vector quantity – is conserved in both the x and y directions.
Kinetic energy — a scalar quantity — is conserved.
This leads to three equations in ten variables: two masses m_1, m_2; and eight speeds v_{1ix}, v_{1fx}, v_{1iy}, v_{1fy}, v_{2ix}, v_{2fx}, v_{2iy}, v_{2fy}.
There are six knowns: the masses m_1, m_2 and the initial speeds $v_{1ix}, v_{1iy}, v_{2ix}, v_{2iy}$.
However, that leaves four unknowns and three equations.

2D Particle Elastic Collision Calculations (cont)

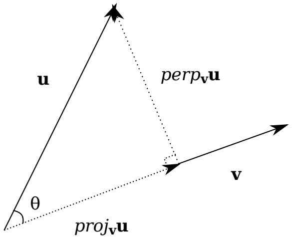
The missing piece is to realise that when the particles collide the only force which operates lies along the line between the centres — which is perpendicular, i.e. normal to both spheres



Thus this is the only direction in which the velocities change, i.e. a 1D problem which we have already solved!
It requires resolving velocity vectors into components along the normal N direction and its perpendicular T.

Vector Resolution

A vector **u** can be resolved into components parallel and perpendicular to another vector **v**.



The vector projection $proj_{\mathbf{v}}\mathbf{u}$ of **u** onto (i.e. parallel to) **v** is

$$proj_{\mathbf{v}}\mathbf{u} = (|\mathbf{u}|cos\theta)\hat{\mathbf{v}} = (\mathbf{u}.\hat{\mathbf{v}})\hat{\mathbf{v}} = (\frac{\mathbf{u}.\mathbf{v}}{|\mathbf{v}|^2})\mathbf{v}$$

The perpendicular $perp_{\mathbf{v}}\mathbf{u}$ of **u** on **v** is

$$perp_{\mathbf{v}}\mathbf{u} = \mathbf{u} - proj_{\mathbf{v}}\mathbf{u}$$

This is one way to solve the 2D (and 3D) sphere collision problem.

Change of Basis: Tangent Space

Another way of thinking about the 2D (and 3D) problem is as a change of basis — using the **TN** (**TBN** in 3D) vectors as the basis. The **TN** basis is referred to as *tangent space* (similar to that used in bump mapping).

2D Particle Inelastic Collision Calculations

For the inelastic 2D case, we solve for the final velocity $\mathbf{v_f}$, which is a vector, by using the 1D equation in (both) the x and y directions v_{fx} and v_{fy} separately.